Abstract: Of all the current methods available to solve the engineering problems using boundary element method, the basic functions which are used to define the unknown quantity in the mathematical formulation, are based on the combination of the shape functions as in a regular finite element procedure. Though this method is popular, it results in a quite complicated numerical procedure especially for thermal radiation problems. Also, since the BEM is based on defining the source density functions, the kernels in the integral equations become singular, strongly singular and hyper singular for which the traditional BEM does not have a common solution procedure. In this research work, it is proposed to develop boundary element solutions for the thermal conduction problems, for three dimensional arbitrarily shaped bodies, using the indirect method i.e. by defining the source density functions. For this purpose, it is proposed to develop new basis functions to solve the integral equations efficiently and effectively. Also the non-uniqueness problem, especially for periodic heating, that may arise in the solution of integral equations will be addressed by combining single layer and double layer formulations. All the numerical procedures developed will be validated against closed form solution or experimental results whichever is available.

Keywords: BEM, basis functions, singular, strongly singular and hyper singular, GF.

1. Introduction
In practice, there are many situations where it is indispensable to know the temperature distribution in a body. For engineering analysis, the most popular computer methods include, for example, the finite difference method (FDM), the finite element method (FEM), and the boundary element method (BEM). The former two, used frequently in numerical modelling, are referred to as domain solution techniques and require full discretization of the whole domain and are often computationally costly and mathematically tricky in the volume mesh generation. As a major complement, the BEM has long been recognized as an efficient numerical tool thanks to its distinguish feature that only the boundary needs to be modelled. Despite this advantage, it is well known as well that the BEM involves quite sophisticated Mathematics and costly numerical integrations. Moreover, meshing the surface of a 3D irregular domain can also be mathematically tricky and computationally expensive [3, 4]. The numerical modelling with boundary elements represents an efficient way to obtain temperature distribution in conductive heat transfer process. The structure of a BEM application

A typical application of BEM consists of the following parts:

- Mathematical model
- Representation formula
• Boundary integral equation
• Boundary elements
• Discrete equations
• Solution of the linear system
• Interpretation

The fundamental solutions traditionally employed in boundary integral analysis for homogeneous materials are ‘free-space’ Green’s Functions (GFs): they satisfy the appropriate differential equation everywhere in space, except at the site where a point load driving force is applied. Derivations for some of the basic GFs can be found in references [5, 6]. Extensions to the case of non-homogeneous materials are reported on references [7-10].

Two-dimensional GF results have appeared in conjunction with a convective heat transfer problem in a homogeneous material [11]. Steady state heat conduction with an arbitrary spatially varying conductivity has recently been investigated [12, 13] and has generated some debate in the literature [14]. A GF for a special type of elastodynamics problem was obtained by Vrettos [15].

Basically finite element method (FEM) requires the subdivision of the region into small elements but BEM on the other hand only requires the subdivision of the boundary of the region (Figure 1.1a, b). BEM consists of two different approaches, the indirect and the direct approach (Figure 1.2a, b). For D-BEM at least one closed boundary is required and one side of the surface can be regarded.

Using I-BEM, both sides of a surface can be considered. I-BEM can also deal with open boundary problems [12].

![Figure 1.1: a) Finite element mesh b) Boundary element mesh](image1)

![Figure 1.2: a) Direct BEM b) Indirect BEM](image2)

Nowadays, BEM is used in many fields; we only research BEM for heat transfer in plate problems.

### 2. Outline of Method of Moments:
Consider the deterministic equation

$$Lf = g$$

(1)

Where \( L \) is a linear operator, \( g \) is a known function and \( f \) is an unknown function to be determined. Let \( f \) be represented by a set of known functions \( f_j, j=1,2,\ldots,N \) termed as basic functions in the domain of \( L \) as a linear combination, given by

$$f = \sum_{n=1}^{N} \beta_j f_j$$

(2)
Where $\beta_i$ are scalar coefficients to be determined. Substituting Eq. 2 into Eq. 1, and using the linearity of $L$, we have

$$\sum_{n=1}^{N} \beta_i L f_j = g$$  \hspace{1cm} (3)

Where the equality is usually approximate. Let $(w_1, w_2, w_3, \ldots)$ define a set of testing functions in the range of $L$. Now, taking the inner product of Eq. 3 with each $w_i$ and using the linearity of inner product defined as

$$\langle f, g \rangle = \int f \cdot g \, ds$$

We obtain a set of linear equations, given by

$$\sum_{n=1}^{N} \beta_i \langle w_i, L f_j \rangle = \langle w_i, g \rangle \quad i = 1, 2, \ldots, N$$  \hspace{1cm} (4)

The set of equations in Eq. 4 may be written in the matrix form as

$$ZX = Y$$  \hspace{1cm} (5)

This can be solved for $X$ using any standard linear equation solution methodologies. The simplicity, accuracy and efficiency of the method of moments lies in choosing proper set of basis/testing functions and applying to the problem at hand.

Using the potential theory and the free space Green’s function, the scattered Thermal potential $T^s$ (Temperature) may be defined as

$$T^s = \int \sigma(r') G(r, r') ds' + \alpha \int \sigma(r') \frac{\partial G(r, r')}{\partial n'} ds'$$  \hspace{1cm} (6)

In the above three equations, $\sigma$ is the source density function independent of $r$ over the surface of the body, $r$ and $r'$ are the position vectors of observation and source points, respectively, with respect to a global co-ordinate system $O$, and $\frac{\partial G}{\partial n'}$ is the normal derivative of Green’s function at source point. Coefficient $\alpha$ is a complex coupling parameter, to be chosen based on the guidelines given by Burton and Miller [Burton and Miller (1971)].

$$k \text{ real or imaginary } \Rightarrow \Im(\alpha) \neq 0$$  \hspace{1cm} (7)

$$k \text{ complex } \Rightarrow \Im(\alpha) = 0$$  \hspace{1cm} (8)

Where $k$ is the propagation vector in the medium.

$G(r, r')$ is the free space Green’s function, given by

$$G(r, r') = \frac{e^{-jkr-r'}}{4\pi |r-r'|}$$  \hspace{1cm} (9)

$G(r, r')$ is the solution of the Helmholtz equation with a point source in homogeneity

$$\left(\nabla^2 + k^2\right)G(r, r') = -\delta(r-r')$$  \hspace{1cm} (10)

And can be interpreted as the solution at the observation point $r$ due to the presence of temperature source of unit strength located at the source point $r'$.

For a rigid body, the normal derivative of total thermal potential (Heat flux), which is the sum of incident and scattered thermal potential, with respect to the observation point on the surface of the body vanishes.

That is

$$\frac{\partial (T^i + T^s)}{\partial n} = 0$$  \hspace{1cm} (11)

$$\frac{\partial T^s}{\partial n} = -\frac{\partial T^i}{\partial n}.$$  \hspace{1cm} (12)

Substituting Eqs. 6, 7 and 8 into Eq. 12,

$$\frac{\partial}{\partial n} \int \sigma(r') G(r, r') ds'$$

$$+ \alpha \frac{\partial}{\partial n} \int \sigma(r') \frac{\partial G(r, r')}{\partial n'} ds' = - \frac{\partial T^i}{\partial n}$$  \hspace{1cm} (13)

Where $n$ and $n'$ are the unit normal vectors at $r$ and $r'$, respectively.
3. Numerical Result and Discussion:

In this section, numerical results for a cube of 1 meter in length, and maintained at a surface temperatures of $100^\circ C$, $200^\circ C$, $500^\circ C$ and $1000^\circ C$ respectively is considered. As an example to show the capability of the BEM which can handle a large number of nodes in the triangulated model, computational domain i.e. a cube is modeled with 866 nodes, 1728 patches, and 2592 edges using triangular patch modeling. Here, in the case of the traditional MoM, one has to invert/solve a matrix of size 866 x 866 for the case of node based basis functions, which is completely eliminated by using the BEM, where only nine basis functions are considered in the cluster. The solution obtained by the cluster of basic functions is further refined five times to get more accurate solution. Figure 3.1 shows the scattering cross section of a thermodynamically rigid cube of length 1m, subjected to heat source with incident plane propagation vector of $k=1$ rad/m.

![Fig. 3.1. Cube with BEM Mesh](image)

**Fig. 3.1. Cube with BEM Mesh**

**Fig. 3.2. Variation of Temperature from the surface of the cube**

In this section, Fig.3.2 shows the temperature gradient for cube whose surface temperature is maintained at $100^\circ C$. To better understand the heat propagation phenomenon from the surface of the cube, it presents a sequence of strong temperature gradient displaying the temperature field along a distance from the surface of the cube. At distance 2m, the temperature gradient is approximately around $25.5^\circ C/m$ and at distance 4m; the temperature gradient is approximately around $12.5^\circ C/m$. This effect is noticeable if one observes the rapid temperature variation in the direction of advance. Similarly as the distance advances, temperature gradient decreases steeply up to 10m and after that till 20m, there is still continuous gradual decrease in the temperature gradient then it attains constant gradient as shown in graph.

3.1. Analytical Solution:

$$T = \text{Surface Temperature} \times \frac{\text{Width of the Cube}}{\text{Distance}}$$

i.e., Temperature at distance

$$T(x) = T_s \times \frac{b}{X}$$

![Fig. 3.3. Variation of Temperature from the surface of the cube](image)

**Fig. 3.3. Variation of Temperature from the surface of the cube**

Again here also i.e. in Fig.3.3 it shows the temperature gradient for cube whose surface temperature is maintained at
200°C. For better understanding of the heat propagation phenomenon from the surface of the cube, it presents a sequence of strong temperature gradient displaying the temperature field along a distance from the surface of the cube. At distance 2m, the temperature gradient is approximately around 120°C/m and at distance 4m; the temperature gradient is approximately around 60.5°C/m. This effect is noticeable if one observes the rapid temperature variation in the direction of advance. Similarly as the distance advances, temperature gradient decreases steeply up to 10m and after that till 20m, there is still continuous gradual decrease in the temperature gradient then it attains constant gradient as shown in graph.

Fig. 3.4. Variation of Temperature from the surface of the cube

Fig.3.4 shows the temperature gradient for cube whose surface temperature is maintained at 500°C. To better understand the heat propagation phenomenon from the surface of the cube, it presents a sequence of strong temperature gradient displaying the temperature field along a distance from the surface of the cube. At distance 2m, the temperature gradient is approximately around 130.3°C/m and at distance 4m; the temperature gradient is approximately around 98.5°C/m. This effect is noticeable if one observes the rapid temperature variation in the direction of advance. Similarly as the distance advances, temperature gradient decreases steeply up to 10m and after that till 20m, there is still continuous gradual decrease in the temperature gradient then it attains constant gradient as shown in graph.

Fig.3.5 shows the temperature gradient for cube whose surface temperature is maintained at 1000°C. To better understand the heat propagation phenomenon from the surface of the cube, it presents a sequence of strong temperature gradient displaying the temperature field along a distance from the surface of the cube. At distance 2m, the temperature gradient is approximately around 600.4°C/m and at distance 4m; the temperature gradient is approximately around 160.6°C/m. This effect is noticeable similarly as the distance advances, temperature gradient decreases steeply up to 10m and after that till 20m, there is still continuous gradual decrease in the temperature gradient then it attains constant gradient as shown in graph.

The variations of temperature for all 4 cases of surface temperature of the cube are represented in the graph and compared with the analytical solutions. All the result obtained in the graph shown above tells us that temperature gradient as we move away from surface of the cube decreases drastically up to 10m depending on the respective boundary value and it becomes stable. From Fig.3.2. to 3.5, it can be seen that the results using the current method are in good agreement with analytical results.

Conclusion:
In this work, a new numerical procedure is developed to generate an almost-diagonal matrix for the solution of boundary integral equation formulation dealing with temperature scattering problems. The major drawback of the traditional boundary integral equation procedure resulting in a dense system matrix is eliminated in this new procedure by grouping the basis functions into a cluster. The geometry of the structure is modeled by planar triangles and the basis functions are defined on the nodes. By doing so, one can get a benefit of a smaller size matrix to begin with. Furthermore, by grouping these node-based basis functions into a cluster, an almost-diagonal matrix is generated. Thus, the solution procedure developed in this work may be utilized for very large scattering problems since the required computer resources are very low. The results computed for these 4 situations (i.e., surface temperature $100^\circ C$, $200^\circ C$, $500^\circ C$, $1000^\circ C$) have shown marked difference in their behavior and the temperature field was found to be strongly dependent on the prescribed boundary value conditions and distances. The solution procedure developed in this work is validated for the scattering cross section of the simple shapes with the closed form solutions wherever available and with the other numerical solution procedures.

References


